On the probability of being synchronizable

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Abstract. We prove that a random automaton with n states and any fixed non-singleton alphabet is synchronizing with high probability. Moreover, we also prove that the convergence speed is exactly $1 - \Theta(\frac{1}{n})$ as conjectured by Cameron [2] for the most interesting 2-letter alphabet case.

1 Synchronizing automata

Suppose $\mathscr{A}$ is a complete deterministic finite automaton whose input alphabet is A and whose state set is Q. The automaton $\mathscr{A}$ is called *synchronizing* if there exists a word $w \in A^*$ whose action *resets* $\mathscr{A}$, that is, w leaves the automaton in one particular state no matter at which state in Q it is applied: $q.w = q'.w$ for all $q, q' \in Q$. Any such word w is called *reset* (or *synchronizing*) for the automaton. The minimum length of reset words for a given automaton $\mathscr{A}$ is called the *reset length* of $\mathscr{A}$ and is denoted by $\mathfrak{C}(\mathscr{A})$.

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applications (coding theory, robotics, testing of reactive systems) and also reveal interesting connections with symbolic dynamics and other parts of mathematics. For a brief introduction to the theory of synchronizing automata we refer the reader to the recent survey [10]. Here we discuss the probability for automata to be synchronizable.

We take an example of such model from [1]. Imagine that you are in a dungeon consisting of a number of interconnected caves, all of which appear identical. Each cave has a common number of one-way doors of different colors through which you may leave; these lead to passages to other caves. There is one more door in each cave; in one cave the extra door leads to freedom, in all the others to instant death. You have a map of the dungeon with the escape door identified, but you do not know in which cave you are. If you are lucky, there is a sequence of doors through which you may pass which take you to the escape cave from any starting point.

The main result of this paper is quite positive; we prove that the probability that if a dungeon has been chosen uniformly at random for a fixed number of colors $t \geq 2$ then there is a life-saving sequence with high probability, namely with the probaility $1 - O(\frac{1}{n})$ where n is the number of caves. Moreover, we prove that the convergence speed is tight for $t = 2$.

This question has been extensively studied and the best result in this direction has been obtained in [12]. They proved that the probability that a 4 letter random automaton is synchronizing is positive and if the number of letters grows together with n as $72\ln(n)$ then a random automaton is synchronizing with high probability.

Let Q stand for $\{1, 2, \ldots n\}$ and Σ_n stand for the probability space of all (unambigious) functions from Q to Q with uniform probability distribution. Let $\mathscr{A} = (Q, \{a, b\})$ be a random automaton whose state set Q equals $\{1, 2, \ldots n\}$, that is a and b are chosen independently from Σ_n.

The *underlying graph* of a given automata $\mathscr{B} = \langle Q, \Sigma, \delta\rangle$ is the digraph whose vertex set is Q and whose edge multiset is $\{(q, \delta(q, a)) \mid q \in Q,\ a \in \Sigma\}$. In other words, the underlying graph of an automaton is obtained by erasing all labels from the arrows of the automaton. An example of automaton and its underlying graph is presented on the figure 1. The underlying graph of a letter $a \in \Sigma$ is the underlying graph of an automaton $\mathscr{B}_a = \langle Q, \{a\}, \delta\rangle$. It is clear that each directed graphs with constant out-degree 1 corresponds to the unique map from Σ_n and thus we can mean Σ_n also as the probability space with uniform distribution on all directed graphs with constant out-degree 1.

We also need a definition of *stable* pairs which was crucial in the solution of the so-called *Road Coloring Problem* [9]. A pair of (different) states $\{p, q\}$ is called stable if for any word u there is a word v such that $p.uv = q.uv$. The stability relation ρ is stable under the actions of all letters and this relation is full whenever $\mathscr{A}$ is synchronizing. The opposite definition is as follows. A pair of states $\{p, q\}$ is called deadlock if there is no word s such that $p.s = q.s$. A subset $A \subseteq Q$ is called F-cliques of $\mathscr{A}$ if it is a maximal by size set such that each pair of states from A is deadlock. It follows from the definition that all F-cliques have the same size.

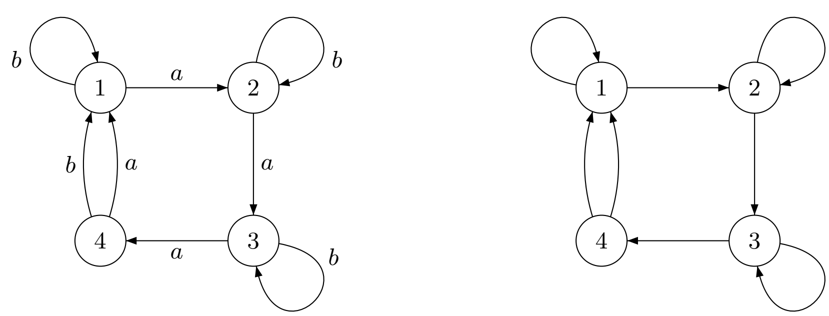

Fig. 1. The automaton $\mathscr{C}_4$ and its underlying graph

Let $\mathscr{A}$ be 2-letter n-state automaton. The scheme of the proof is as follows.

Suppose $\mathscr{A}$ is not synchronizing; then there is an F-clique F_0. The number of states accessible from F_0 is at least $n/4e^2$ with probability $1 - O(1/n)$. Indeed, the probability that there is a closed subset in automaton of size less than $n/4e^2$

is bounded by

$$\sum_{i=1}^{n/4e^2} \binom{n}{i} (\frac{i}{n})^{2i} \leq (\frac{ei}{n})^i.$$

For $i \leq n/4e^2$ we get that $(\frac{ei}{n})^i \geq 2(\frac{e(i+1)}{n})^{i+1}$. Hence the sum can be bounded by double first element $2(\frac{e}{n})$. Take an arbitrary state $p \in F_0$. The probability that p reaches less than $n/4e^2$ states is at most $1 - O(1/n)$.

In section 2 we prove that for any $c > 0$ the underlying graph of any letter has a unique highest tree and that this tree is higher than all other trees by c with probability $1-\Theta(c/\sqrt{n})$ and thus the probability for automata to have such tree for at least one letter (say a) is $1-\Theta(c^2/n)$. Note that the states of this tree (in particular, the set H of states which higher in the highest tree than states of other trees by $c/2$) are random for letter b. We want to prove that the subset H is *reachable* from F_0 in automata with high probability, that is for some state $s \in F_0$ there is a word u which maps s to some state of H. If this subset would be random for the automaton it would be trivial.

Using that for a random tree T of height h the number of states of level $t \leq h, t = \Theta(h)$ is linear on t with high probability we prove that that there are at least $d = 8 \ln n$ states in H.

Since H is random for b there is a set H_b with at least $d/2$ states incoming to distinct states of H by letter b with probability $1 - o(1/n^2)$. because the probability of the complement event can be bounded by

$$\sum_{k=d/2}^{n} P(|Q.b| = n-k)(2^d \binom{k}{d/2} / \binom{n}{d/2}) \leq \sum_{k=d/2}^{n} \frac{\binom{n}{k} S(n, n-k)(n-k)!}{n^n} 2^d (\frac{k}{n})^{d/2},$$

where $S(n, n-k)$ is the Stirling number of the second kind. Indeed, for every possible size of $Q.b$ we choose a subset of at least $d/2$ states in H in at most 2^d ways and then the probability that this subset does not lie in $Q \setminus Q.b$ is equal $\binom{k}{d/2}/\binom{n}{d/2}$. Then we choose $Q.b$ by $\binom{n}{k}$ ways, define partition of Q in $n-k$ non-empty subsets and finally determine bijection between these subsets and $Q.b$ in $(n-k)!$ ways.

Hence all states of H_b has different images we can randomly add arbitrary number of states with the same images to get random set for b or equivalently suppose that the set H_b is already random for letter b. Thus the set H_b is random for each letter independently but can depend on its combination. This problem can be solved as follows. Take a random letter b with at least $d/4$ states H_1 incoming H by letter b, choose uniformly at random a subset H_2 of size $d/4$ and redefine outgoing edges uniformly at random to H. Then we get uniformly at random an automaton $\mathscr{B}$ with at least $d/2$ states incoming H by letter b, that is the automaton like $\mathscr{A}$. The probability that a subset H_b is reachable from F_0 in this automaton is at least the probability that a random subset of $d/4$ states are reachable from F_0. The probability of the complement event can be bounded by $\binom{(1-1/4e^2)n}{d}/\binom{n}{d} \leq (\frac{1-1/4e^2}{1-0.99(1-1/4e^2)})^{8\ln n} = o(1/n^2)$.

Since H_b is reachable from F_0 with probability $1-o(1/n^2)$, H is also reachable from F_0 with probability $1 - o(1/n^2)$. Thus with probability $1 - O(1/n)$ there is a state s reachable from F-clique F_0 in the automaton $\mathscr{A}$ such that its level (in the digraph of letter a) is higher than levels of states from other trees by $c/2$. Using this and ideas from [9] in section 3 we prove that $\mathscr{A}$ has at least $c/2$ pairwise distinct stable pairs which are random for letter b.

Since these pairs are random for letter b it is easy to prove that there are $n^{1/3}$ pairwise distinct stable pairs of states random for letter a with probability $1 - O(1/n)$.

Using that the number of cycles of a random digraph is strongly concentrated near $\ln n/2$ we shall get that a has at most $\ln^2 n$ cycles with probability $1 - O(1/n)$.

The cycle vertices of a induce the state partition as follows. Two states p, q are in the same class if and only if for some positive integer $d > 0$ we have $p.b^d = q.b^d$. Thus for any cluster C_1 with cycle of length l all states in C_1 are partitioned in $l = O(\sqrt{n})$ classes. Then with high probability for each pair of (probably equal) *big* clusters C_1, C_2 such that $|C_1|, |C_2| > n^{1/6}$ there are a lot of stable pairs of states p_1, p_2 such that p_i lies in the cluster of C_i (for $i \in \{1, 2\}$) and p_1 and p_2 from different classes. Using that there are a lot of such pair for the case when clustes coincide we get that all pairs from the same clusters are stable, and using that there are pairs for each pairs of big clusters we get that all pairs from different clusters are also stable with probability $1 - O(1/n)$. This would immediately implies that all pairs from big clusters are synchronizing with probability $1 - O(1/n)$ because they can be mapped to some cycle pair. Since there can be only few small clusters we get that there are at least $n-o(n)$ pairwise synchronizing states with probability $1-O(1/n)$. This would contradicts the fact that F_0 reaches $n/4e^2$ different states with probability $1 - O(1/n)$.

2 A probability to have a unique highest tree

Theorem 1. *For $c = o(\sqrt{n})$ the probability that a random digraph from Σ_n has a unique highest tree and that this tree is higher than all other by c is equal $1 - \Theta(c/\sqrt{n})$.*

We prove that the probability of negation event tends to 0, i.e. we prove that the probability that a random digraph g has at least two trees of height greater than height of g minus c tends to 0 as $\Theta(1/n\sqrt{n})$. We almost repeat the proof of the limit distribution for the height of a random digraph from Σ_n from [6].

We need the following additional notations from [6]:

- $F_{n,N}$ denotes the probability space on the set of all (ordered) forests with $n + N$ states and N root vertices and uniform probability distribution.
- For $x \in F_{n,N}$ or $x \in \Sigma_n$ denote by $\tau(x)$ the height of x.
- G^N denote the probability space of random *critical branching processes*, for $\mu \in G^N$ and $t \geq 0$ by $\mu(t)$ we denote the number of elements in this process at moment t, $\mu(0) = N$ and by $\nu(\mu)$ denote the number of elements in μ after the degeneration moment $\tau(\mu)$ of the process μ.

– Given a $\mu \in G^N$ and $i \in \{1, 2, \ldots N\}$ denote by ν^i the number of elements of i-th tree from μ, by ν_t^i the value of ν^i under the condition $\tau(\nu^i) < t$, i.e. $P(\nu_t^i = k) = P(\nu^i = k \mid \tau(\nu^i) < t)$ and correspondingly by $\bar{\nu}_t^i$ the value of ν^i under the condition $\tau(\nu^i) = t$.

Theorem 2 (Kolchin [6]). *For any $x > 0$*

$$P_{g \in \Sigma_n}(\tau(g)/\sqrt{n} \leq x) \to \sum_{k=-\infty}^{+\infty} (-1)^k e^{-k^2 x^2/2}.$$

It follows from this theorem that for a random digraph $g \in \Sigma_n$ the value of $\tau(g)$ has order $\sqrt{n}$, namely

$$P_{g \in \Sigma_n}(\frac{\tau(g)}{\sqrt{n}} \in (r_1, r_2)) = 1 - \delta(r_1, r_2) \tag{1}$$

where $\delta(r_1, r_2)$ is arbitrary small for $r_1, 1/r_2$ small enough.

Actually, the same is true for the number of rooted vertices $\lambda(g)$.

$$P_{g \in \Sigma_n}(\frac{\lambda(g)}{\sqrt{n}} \in (r_1, r_2)) = 1 - \delta(r_1, r_2) \tag{2}$$

where $\delta(r_1, r_2)$ is arbitrary small for $r_1, 1/r_2$ small enough.

It is clear that any digraph $g \in \Sigma_n$ with $\lambda(g) = N$ corresponds to the unique forest $f(g) \in F_{n-N,N}$ of its trees. Hence

$$P_{g \in \Sigma_n}(\tau(g) = k \mid \lambda(g) = N) = P_{f \in F_{n-N,N}}(\tau(f) = k).$$

This implies that the height distribution for Σ_n can be averaged by λ of the height distribution on $F_{n,N}$. In one's turn the height distribution on $F_{n-N,N}$ can be considered as the distribution of the degeneration moment on G^N under condition that there are n elements after degeneration, namely,

$$P_{f \in F_{n-N,N}}(\tau(f) = k) = P_{\mu \in G^N}(\tau(\mu) = k + 1 \mid \nu(\mu) = n).$$

Slightly overload notations denote by $f(\mu)$ the forest we get after degeneration of μ. Denote by $A_{N,t} \subseteq G^N$ the subset of $\mu \in G^N$ such that $f(\mu)$ has at least one tree of height t, at least one tree of height between t and $t + c$ and all other trees are lower than $t + 1$.

We now briefly describe the changes we have to do in the proof from [6]. For a moment we consider only random branching processes $\mu \in G^N$ and thus will omit it as index of P.

Lemma 1 (Kolchin [6]). *If $P(\nu(\mu) = n + N) > 0$ then for any $t > 0$*

$$P(\tau(\mu) \leq t \mid \nu(\mu) = n + N) = (P(\tau(\nu_1) \leq t)^N) \frac{P(\nu_{N,t} = n + N)}{\nu_N = n + N}$$

where $\nu_{N,t} = \sum_{i=1}^N \nu_t^i$ and $\nu_N = \sum_{i=1}^N \nu^i$.

If we replace $P(\tau(\mu) \leq t \mid \nu(\mu) = n+N)$ in the above lemma by $P(A_{N,t} \mid \nu(\mu) = n+N)$ we get

$$P(A_{N,t} \mid \nu(\mu) = n+N) = \Theta(1)cN(N-1)P(\tau(\nu^{N-1}) = t)P(\tau(\nu^N) \in [t, t+c])* \\ * (P(\tau(\nu_1) \leq t)^{N-2})\frac{P(\nu_{N-2,t} = n + N - \bar{\nu}^{N-1} - \bar{\nu}^N)}{P(\nu_N = n + N)}.$$

Indeed, we choose tree of height t by N ways, then choose tree and its height between t and $t+c$ by $(N-1)$ ways and replace it with $\bar{\nu}^i_t$. Since all one-element processes of μ are independent and uniformly distributed we can choose numbers $N-1, N$ and then multiply by $N(N-1)$.

Using that in the theorem $N \to \infty$ and $P(\tau(\nu^i) = t) = 2/t^2(1+o(1))$ we can simplify this formulae as

$$P(A_{N,t} \mid \nu(\mu) = n + N) = \Theta(1)cN^2/t^4(P(\tau(\nu_1) \leq t)^N)\frac{P(\nu_{N,t} = n + N)}{\nu_N = n + N}.$$

We have erased $-\bar{\nu}^{N-1} - \bar{\nu}^N$ because the probability that $\bar{\nu}^i > \sqrt{N}$ also tends to 0 as $N \to \infty$. Thus we have got the same formulae at the right with an additional factor $\Theta(1)cN^2/t^4$.

Given $x > 0$ let $B_{n,N,x} \subseteq F_{n,N}$ be the set of all forests f of height at most $x\sqrt{n}$ such that there are at least two trees of height $\tau(f) - c$. Then replacing event $\tau_{F_{n,N}} \leq x\sqrt{n}$ with $B_{n,N,x}$ we get the following analog of Theorem 3 of Kolchin [6].

Theorem 3. *For any $x > 0$ and $n, N \to \infty$ such that $0 < z_1 < z = N/\sqrt{n} < z_1 < \infty$ we get*

$$P_{f\in F_{n,N}}(B_{n,N,x}) = \Theta(1)(cz^2/x^4n)\frac{e^{z^2/2}}{z\sqrt{2\Pi}}\int_{-\infty}^{+\infty} e^{-i\nu - zf(x,\nu)/x}d\nu(1 + o(1)).$$

Finally, denote by $B_{n,x}$ the set of all digraph g in Σ_n such that $f(g) \in B_{n,\lambda(g),x}$ we get

$$P_{g\in\Sigma_n}(B_{n,x}) = \ldots = (c\Theta(1)/x^4n)\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty}\int_0^{+\infty} z^2e^{-i\nu - zf(x,\nu)/x}dzd\nu$$

that is we have an additional factor $\Theta(1)c/x^4n$ and an additional factor z^2 in the integral. Using that

$$\int_0^{+\infty} z^2e^{-i\nu - zf(x,\nu)/x}dz = \frac{e^{-i\nu}x}{f(x,\nu)}\int_0^{+\infty}\alpha^2e^{-\alpha}d\alpha = \\ = \frac{e^{-i\nu}x}{f(x,\nu)}e^{-\alpha}(\alpha^2 - 2\alpha + 2)|^0_\infty = \frac{2e^{-i\nu}x}{f(x,\nu)}$$

we get the same expression as a result with the factor $\frac{\Theta(1)c}{x^4 n}$ and thus eventually we obtain that

$$P_{g\in\Sigma_n}(B_{n,x}) = \frac{\Theta(1)c}{x^4 n}\sum_{k=-\infty}^{+\infty}(-1)^k e^{-k^2x^2/2}.$$

Since $\sum_{k=-\infty}^{+\infty}(-1)^k e^{-k^2x^2/2}$ is a limit of probability $P_{g\in\Sigma_n}(\tau(g)/\sqrt{n}\leq x)$ which is positive constant we get

$$P_{g\in\Sigma_n}(B_{n,x}) = \frac{\Theta(1)c}{x^4 n}.$$

Denote by $B_n \subseteq \Sigma_n$ the set of all digraphs g such that $f(g)\in B_{n,\lambda(g),x}$ for some x. Replacing t with $x\sqrt{n}$ we get

$$P_{g\in\Sigma_n}(B_n) = \lim_{d\to 0}\sum_{t=[d\sqrt{n}]}^{[\sqrt{n}/d]} P_{g\in\Sigma_n}(B_{n,\lambda(g),t/\sqrt{n}}) =$$

$$= \lim_{d\to 0}\sum_{t=d\sqrt{n}}^{\sqrt{n}/d}\frac{\Theta(1)cn}{t^4}o(1+o(1)) = \lim_{d\to 0}\frac{\Theta(1)cn}{t^4}\int_{d\sqrt{n}}^{\sqrt{n}/d}\frac{dt}{t^4} = o(c/\sqrt{n}).$$

Thus the theorem is proved. $\square$

Since the letters for random automata are chosen independently the following corollary is straightforward.

Corollary 1. *For any integer constant $c = o(\sqrt{(n)})$ the probability that a random t-letter automaton with n states has a unique highest tree for at least one letter and that this tree is higher than all other trees of this letter by c is equal $1-\Theta(c^{2t}/\sqrt{tn})$.*

3 Searching for stable pairs

We need following definitions from [9]. A pair of states $\{p, q\}$ is called deadlock if there is no word s such that $p.s = q.s$. A subset $A \subseteq Q$ is called F-cliques of $\mathscr{A}$ if it is a maximal by size set such that each pair of states from A is deadlock. It follows from the definition that all F-cliques have the same size. Let us reformulate Lemma 2 in the following way.

Lemma 2 (Trahtman [9]). *Let A and B be two distinct F-cliques such that $A\setminus B = p, B\setminus A = q$ for some pair of states p, q; Then p, q is a stable pair.*

Proof. Arguing by contradiction, suppose there is a word s such that $\{p.s, q.s\}$ is deadlock. Then $(A\cup B).s$ is an F-clique because all pairs are deadlock and $|(A\cup B)| = |A|+1 > |A|$ because $p.s \neq q.s$. This contradicts the maximality of A.

Notice that the condition that the underlying graph of the automation is AGW graph is not used in Theorem 2 from [9], this property is only used to prove that the underlying graph of letter has a unique highest tree. We only need in Theorem 2 from [9] that states from highest tree was reachable from some F-cliques.

Theorem 4. *The probability that a random* 2*-letter automaton have* $k = n^{1/3}$ *stable pairs is* $1 - O(1/n)$.

Proof. By Corollary 1 we get that there is a letter (say a) in the automaton $\mathscr{A}$ with highest tree T which is higher than all other trees by constant c with probability $1 - O(1/n)$. By Lemma 2 there is a stable pair in this case and we can consider the underlying graph of letter a in the factor automaton by the stability relation. Since the relation is stable the heights of corresponding trees in the factor automaton $\mathscr{B}$ can not increase and if the height of T the factor automaton is decreased by r then there are at least r stable pairs (connecting states of different levels) in $\mathscr{A}$. Thus there are at least r stable pairs and if $r < c$ then the highest tree in $\mathscr{B}$ is higher than all other trees by $c - r$. Hence by induction we get that $\mathscr{B}$ has at least $c - r$ stable pairs.

Since these c stable pairs are random for b each pair p, q is merged and does not intersect with other pairs by b with probability $1 - c/n$ in one step, the probability that p, q is not merged and does not intersect with other pairs for $n^{1/3}$ steps is $1 - 1/n^{1/3}$. Hence for $c = 9$ the probability that there is such a pair is $1 - O(1/n)$. Thus the theorem is proved. $\square$

4 Reachable and strongly connected automata

Theorem 5. *The probability that* 2*-letter random automaton* $\mathscr{A}$ *with* n *states is not reachable is at least* $\Theta(1/n)$.

Proof. Let $\mathscr{A}$ has exactly one *disconnected* state, that is the singleton weakly connected component. These automata can be counted as follows. We first choose a disconnected state in n ways (the transitions for this state is defined in 1 way) and then the number of different transitions for any other state is

$$1(n-2) + (n-2)(n-1) = n(n-2)$$

because other states are not disconnected. Thus the probability of such automata is equal

$$\frac{n(n(n-2))^{n-1}}{n^{2n}} = \frac{1}{n}(1 - \frac{2}{n})^{n-1} = \Theta(1/n).$$

Thus the theorem is proved.

Hence synchronzing automata is necessarly reachable we get that the main result.

Theorem 6. *The probability that a* 2*-letter random automaton with* n *states is synchronizing tends to* 1 *as* $1 - \Theta(1/n)$.

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